REDUCTION OF THE EFFECTIVE MASS OF THE REISSNER-NORDSTRÖM SPACETIME

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Abstract

We derive the Teixeira, Wolk and Som method [1], for obtaining electrostatic solutions from given vacuum solutions, in its inverse form. Then we use it to obtain the geometrical mass M_S in the Schwarzschild spacetime, and we find $M_S^2 = M^2 - Q^2$, where M and Q are, respectively, the mass and charge parameters of the Reissner-Nordström spacetime. We compare M_S to the corresponding active gravitational mass and mass function.

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1 Introduction

For a bounded spherically symmetric static distribution of matter the total mass is well defined. We know that outside of the distribution the spacetime must be described by the Schwarzschild spacetime, therefore it follows from the junction conditions [2] that the total mass of the system is equal to the Schwarzschild mass parameter [3]. However the definition of the total effective mass content within a given spherical surface inside a non vacuum spacetime is not unique. This ambiguity has been the subject of long discussions, giving rise to different definitions of energy (see references given in [4])

Our aim in this article is to revisit the question of effective mass in the Reissner-Nordström (RN) spacetime.

The study of charged bodies in Einstein's theory contributes to better understand the structure of spacetime, as it is showed by many new solutions recently studied for electrovacuum (see [5, 6] and references therein) and different charged sources (see [7, 8] and references therein). For the spherically symmetric static spacetime the solution of the coupled Einstein and Maxwell equations is the RN solution. This solution is the unique black hole solution with a regular event horizon and assymptotically flat behaviour. The RN spacetime provides a more general framework to study the structure of Schwarzschild spacetime. The fact that RN solution has two horizons, an external event horizon and an internal Cauchy horizon, provides a convenient bridge to the study of the Kerr solution, as pointed out by Chandrasekhar [9]. Furthermore, the RN field can be used as a simple model of the electron, as suggested by Bonnor and Cooperstock [10]. We point out too, that the study of the effective mass in the RN spacetime might help to find the corresponding one in the Kerr spacetime.

To study the effective mass in RN spacetime we determine the corresponding Schwarzschild spacetime, henceforth, the Schwarzschild mass parameter, thus obtained, corresponds to the so called geometrical mass in the RN spacetime. In order to obtain this correspondence we study a class of solutions where we impose that the metric component g_{tt} is functionally related to the electrostatic potential ϕ. This technique of determining solutions of Einstein and Maxwell equations is not new. It started with a remarkable paper by Weyl in 1917 [11], where he found a class of electrostatic cylindrically symmetric solutions by imposing that $g_{tt}(\phi)$. Majumdar in 1947 [12] generalized this result to systems without spatial symmetry. The inclusion of solutions with magnetostatic fields came through the works of Papapetrou in 1947 [13] and Bonnor in 1954 [14]. But in 1955 Ehlers [15] gave a new approach to generate solutions of Einstein and Maxwell equations by starting from given vacuum solutions. Later, in the same vein, other solutions were found by Bonnor in 1961 [16] and Janis, Robinson and Winicour in 1967 [17]. However, all these solutions have the handicap of not switching back, in a simple way, to its original vacuum solution. This difficulty was overcomed in 1976 by Teixeira, Wolk and Som (TWS). By using the operation of duality rotation they were able to introduce simultaneously electrostatic and magnetostatic fields, having the feature that by a proper choice of the constants the original vacuum solution emerged in a straightforward way. This work generalizes the previous results [15, 16, 17]. In 1977 Som, Santos and Teixeira [18] applied the TWS method to obtain the RN solution from the Schwarzschild solution. This result was reobtained recently [19].

The plan of the paper is as follows. In section 2 we derive the method developed by TWS [1] in its inverse form, i.e., given a solution of the Einstein and Maxwell equations how to find its corresponding vacuum solution. Next, we apply the inverse method to obtain the corresponding Schwarzschild spacetime to the RN spacetime. By doing this we deduce the geometrical mass in the RN spacetime. In sections 3 and 4 we obtain the active gravitational mass and mass function in the RN spacetime and compare to the results of section 2. There is a discussion in the last section.

2 The geometrical mass

Here we present a method for deriving a class of vacuum solutions of Einstein's equations out of a given solution of Einstein and Maxwell equations. The class of solutions that we search for is when we impose that the g_{tt} metric component of the vacuum solution is functionally related to the same component of the electrovacuum solution. This method is the inverse of the TWS method [1]. Then we apply this method to deduce the Schwarzschild solution from the RN solution.

2.1 The inverse of the TWS method

We consider the static line element corresponding to the electrovac solution

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}h_{ij}dx^i dx^j, \tag{1}$$

where ψ and h_{ij} are functions of the spatial components x^k (latin indices run from 1 to 3). The Maxwell's equations in empty space read

$$F^{\mu\nu}{}_{;\mu} = 0, \tag{2}$$

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu;\rho} = 0, \tag{3}$$

with $\epsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric tensor (greek indices run from 0 to 3) with convention $\epsilon^{0123} = 1$. From (1) and (3) the electromagnetic field tensor $F_{\mu\nu}$ can be written with the static non null components

$$F_{0i} = -\phi_{,i}, \tag{4}$$

where ϕ is the electrostatic potential. Substituting (1) and (4) into (2) we obtain

$$(h^{1/2}e^{-2\psi}h^{ij}\phi_{,i})_{,j} = 0, \tag{5}$$

with h being the determinant of the spatial metric h_{ij}. Einstein electrovac equations read

$$G_{\mu\nu} = \kappa E_{\mu\nu} = \kappa \left(g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \tag{6}$$

where $G_{\mu\nu}$ is the Einstein tensor and $E_{\mu\nu}$ is the electromagnetic energy tensor. Substituting (1) and (4) into (6) gives

$$e^{2\psi} h^{-1/2} (h^{1/2} h^{ij} \psi_{,i})_{,j} = h^{km} \phi_{,k} \phi_{,m}, \tag{7}$$

$$H_{ij} + 2\psi_{,i}\psi_{,j} = -2e^{-2\psi} \phi_{,i}\phi_{,j}, \tag{8}$$

where H_{ij} is the Ricci tensor in the 3 dimensional space.

If we consider now the static line element representing a solution of the Einstein equations in the vacuum:

$$ds^2 = e^{2V} dt^2 - e^{-2V} h_{ij} dx^i dx^j, \tag{9}$$

with V function of x^k, the corresponding Einstein equations read:

$$(h^{1/2} h^{ij} V_{,i})_{,j} = 0, \tag{10}$$

$$H_{ij} + 2V_{,i}V_{,j} = 0. \tag{11}$$

We can then formulate the inverse of the TWS problem. Starting from a static solution (ψ, h_{ij}, ϕ) of Maxwell equations (5) and Einstein electrovac equations (7) and (8), we want to obtain the corresponding Einstein vacuum solution (V, h_{ij}). We then search for a class of solutions where V is functionally related to ψ.

From (5) and (10), we obtain

$$aV_{,i} = e^{-2\psi} \phi_{,i}, \tag{12}$$

where a is an integration constant. Substituting (11) into (8) and considering (12), we get

$$V_{,i} = \left(1 + a^2 e^{2\psi}\right)^{-1/2} \psi_{,i}. \tag{13}$$

By integration, (13) leads to

$$e^{2V} = e^{2\psi} \left[\frac{1 + (1 + a^2)^{1/2}}{1 + (1 + a^2 e^{2\psi})^{1/2}} \right]^2, \tag{14}$$

where we have chosen the integration constant such that when $a = 0$, the electric field is zero and the electrovac solution (1) reduces to the vacuum solution (9). Introducing (12) into (7) with (13) yields

$$h^{-1/2} (h^{1/2} h^{ij} \psi_{,i})_{,j} = \frac{a^2 e^{2\psi}}{1 + a^2 e^{2\psi}} h^{km} \psi_{,k} \psi_{,m}, \tag{15}$$

which means that the constant a is determined by the parameters involved in ψ and h_{ij}.

2.2 Schwarzschild solution from a RN solution

We now apply the preceding method to determine the Schwarzschild solution associated with a RN metric. First, substituting the RN line element,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{16}$$

into (15) we find for a

$$aM_S = Q, \tag{17}$$

where

$$M_S = (M^2 - Q^2)^{1/2}. \tag{18}$$

We see from (17) that when $a = 0$ then $Q = 0$, reducing the RN spacetime (16) to the vacuum Schwarzschild spacetime. The term $e^{2\psi}$ in (1) can be expressed from (16), with the aid of (18), as

$$e^{2\psi} = \frac{(r - M_S - M)(r + M_S - M)}{r^2}. \tag{19}$$

Substituting (17) and (19) into (14) we obtain

$$e^{2V} = 1 - \frac{2M_S}{r + M_S - M}. \tag{20}$$

Considering (20), the vacuum solution (9) becomes

$$ds^2 = \left(1 - \frac{2M_S}{R}\right) dt^2 - \left(1 - \frac{2M_S}{R}\right)^{-1} dR^2 - R^2 d\Omega^2, \tag{21}$$

where R is given by

$$R = r + M_S - M. \tag{22}$$

Hence, the corresponding vacuum solution to the RN spacetime is the Schwarzschild spacetime (21) with geometrical mass $M_S = (M^2 - Q^2)^{1/2}$ and with a scale shift (22) in the radial coordinate R.

The event horizon r_{eh} and the Cauchy horizon r_{Ch} for the RN spacetime (16) are then

$$r_{eh} = M + M_S, \quad r_{Ch} = M - M_S, \tag{23}$$

while for the corresponding Schwarzschild spacetime (21) these two surfaces represent, respectively, the event horizon R_{eh} and singularity R_0,

$$R_{eh} = r_{eh} + M_S - M = 2M_S, \quad R_0 = r_{Ch} + M_S - M = 0. \tag{24}$$

3 The active gravitational mass

The active gravitational mass density μ, given by Tolman [20] and Whittaker [21], is for the field equations (6)

$$\mu = E^0{}_0 - E^i{}_i, \tag{25}$$

and the total active gravitational mass within a volume V, coming out from the work of Whittaker [21] reads

$$M_a = \int_V \mu(-g)^{1/2} dx^1 dx^2 dx^3, \tag{26}$$

where g is the four dimensional determinant of the metric. Applying (26) to the RN spacetime (16) we find

$$M_a(\infty) - M_a(r) = \int_r^\infty \frac{Q^2}{r^2} dr, \tag{27}$$

which, assuming that $M_a(\infty) = M$, leads to

$$M_a(r) = M - \frac{Q^2}{r}. \tag{28}$$

The active mass is positive or null for $r \geq r_0 = Q^2/M$ with $r_{Ch} \leq r_0 \leq r_{eh}$. It can take negative values for $r < r_0$ and this possibility has been discussed in a number of papers [10, 22, 23, 24]. The active gravitational mass (28) for the event horizon and Cauchy horizon are, respectively,

$$M_a(r_{eh}) = M_S, \quad M_a(r_{Ch}) = -M_S. \tag{29}$$

The spacetime external to $r = r_{eh}$ is very similar to the Schwarzschild spacetime external to the surface $r = 2M_S$ (see discussion in [9] p. 209). Hence the surface $r = r_{eh}$ is an event horizon in the same sense that $r = 2M_S$ in Schwarzschild spacetime. However, the spacetime internal to $r = r_{eh}$ has a completely different structure as compared to the Schwarzschild spacetime (refer again to [9]). It is interesting that the active gravitational mass (28) measures in RN spacetime the corresponding Schwarzschild mass at $r = r_{eh}$.

The electric field E due to (16) is

$$E = \frac{Q}{r^2}, \tag{30}$$

and from (5) the corresponding non-relativistic Maxwell energy, $M_E(r, \infty)$, entrapped outside the spherical surface of radius r is given by

$$M_E(r, \infty) = \frac{1}{2} \int_r^\infty E^2 r^2 dr. \tag{31}$$

Using (30) and (31), the active gravitational mass (28) can be rewritten

$$M_a(r) = M - 2M_E(r,\infty). \tag{32}$$

From (32) we have that $M_a(r)$ is the total active gravitational mass minus twice the mass equivalent to the non-relativistic energy stored by the electric field E outside the spherical surface of radius r.

Calculating the circular geodesics in the equatorial plane and the radial geodesics of RN spacetime for a chargeless particle we obtain, respectively, from (16) and (28),

$$\frac{d^2r}{d\tau^2} = -\frac{1}{r^2}\left(M - \frac{Q^2}{r}\right) = -\frac{M_a(r)}{r^2}, \quad \left(\frac{d\phi}{d\tau}\right)^2 = \frac{1}{r^3}\left(M - \frac{Q^2}{r}\right) = \frac{M_a(r)}{r^3}, \tag{33}$$

where τ is the proper time. The motion of a chargeless particle will then be affected by the charge of the black hole eventually producing repulsive forces for sufficiently small values of r. The active gravitational mass casts locally the equations of motion in a Newtonian like form.

4 The mass function

There exists a further different way to define the gravitational mass of a system. If we match a spherical distribution of matter to the exterior Schwarzschild spacetime we obtain at the surface of discontinuity the Schwarzschild mass being equal to $rR^{\phi}{}_{\theta\phi\theta}/2$ where $R\phi_{\theta\phi\theta}$ is a Riemann tensor component. The interpretation of this mass as the total mass inside the sphere suggests that the total mass entrapped inside a sphere of radius r, called the mass function, may be defined by

$$M_f(r) = \frac{1}{2} r R^{\phi}{}_{\theta\phi\theta}. \tag{34}$$

This definition has been first considered by May and White [25] and since currently used in gravitational collapse [26, 27, 28, 29, 30]. For RN spacetime (16) we have for (34)

$$M_f(r) = M - \frac{Q^2}{2r}, \tag{35}$$

where

$$M = \frac{1}{2} r C^{\phi}{}_{\theta\phi\theta}, \tag{36}$$

$C^{\phi}{}_{\theta\phi\theta}$ being a Weyl tensor component. M is called the pure gravitational mass, since it arises only from the Weyl tensor. The mass function is always positive or null for $r \geq r_1 = Q^2/2M$ with $r_1 = r_0/2$ inside the event horizon. Negative values for $M_f(r)$ for $r < r_1$ are considered in [31]. The mass function (35) at the event horizon and Cauchy horizon (23) are, respectively,

$$M_f(r_{eh}) = \frac{1}{2}(M + M_S), \quad M_f(r_{Ch}) = \frac{1}{2}(M - M_S). \tag{37}$$

Considering (30) and (31) we have

$$M_f(r) = M - M_E(r, \infty), \tag{38}$$

which can be compared to (32).

5 Discussion

We have presented the TWS method [1] in its inverse form and applied it to obtain the vacuum Schwarzschild solution from the electrovacuum RN solution. The vacuum metric thus derived has a geometrical mass M_S (17). Then we calculate for RN spacetime its active gravitational mass (28) and mass function (35). Both results are equal to M asymptotically but, in general, differ. This illustrates the ambiguity in the localization of energy. We compare $M_a(r)$ and $M_f(r)$ with M_S at the horizons, respectively (29) and (37), and obtain that $M_a(r_{eh}) = M_S$. This result is not so surprising because of the following reasons. The structure of the event horizon for RN spacetime and Schwarzshild spacetime are similar [9]. The equations of motion (33) are cast in a locally Newtonian like form with the aid of the active gravitational mass. Furthermore, Herrera and Santos [4] by analysing the energy content of a slowly collapsing gravitating sphere conclude that the active gravitating mass grasps better the physical content of matter than the mass function. Hence we can say that our results give further support the physical meaningfulness of $M_a(r)$.

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